

Noncommutative pfaffians and representations. Applications to classification of states of five-dimensional quasi-spin. *

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Noncommutative pfaffians associated with an orthogonal algebra are some special elements of the universal enveloping algebra. It is proved that in the case $N = 2n + 1$ some of these pfaffians, denoted as $PfF_{\widehat{-n}}$ and $PfF_{\widehat{n}}$, act on the space of \mathfrak{o}_{2n-1} -highest vectors of a \mathfrak{o}_{2n+1} -representation. There exist the Mickelsson-Zhelobenko algebra of raising operators $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ which naturally acts on this space. We find explicitly an element of the Mickelsson-Zhelobenko algebra, which acts on this space in the same way as the pfaffian $PfF_{\widehat{n}}$. As a by-product we find explicit formulae for the action of the pfaffian $PfF_{\widehat{n}}$ in the Gelfand-Tsetlin-Molev base. The action of pfaffians in the tensor realization of representation is considered in the appendix.

1 Introduction

In the paper we study the noncommutative pfaffians, which are some special elements of the universal enveloping algebra $U(\mathfrak{o}_N)$. The main subject is an investigation of an action of noncommutative pfaffians in representations in the case of odd N .

First it is noted that all pfaffians for map weight vectors to weight vectors and the weight changes by a simple rule.

Then it is noted that in the case $N = 2n + 1$ some pfaffian (these pfaffians are denoted as $PfF_{\widehat{-n}}$ and $PfF_{\widehat{n}}$, the explanation of these notations see below in the introduction) commute with the subalgebra $\mathfrak{o}_{2n-1} \subset \mathfrak{o}_{2n+1}$. Thus these pfaffians act on the space of \mathfrak{o}_{2n-1} -highest vectors of a \mathfrak{o}_{2n+1} representation.

Thus the pfaffians $PfF_{\widehat{-n}}$, $PfF_{\widehat{n}}$ act as raising operators in the problem of the construction of a bases of a \mathfrak{o}_{2n+1} -representation of Gelfand-Tsetlin type.

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The problem is to construct a bases in a \mathfrak{o}_{2n+1} -representation and for different n the constructions must be coherent. This means that the bases of a \mathfrak{o}_{2n+1} -representation must be a union of bases in \mathfrak{o}_{2n-1} -representation into which the \mathfrak{o}_{2n+1} -representation splits.

There papers where such a base is constructed in the simplest nontrivial case $\mathfrak{o}_3 \subset \mathfrak{o}_5$ ([6],[7],[8],[9]).

In general case, using ideas of Gelfand and Tsetlin, such a base was constructed by Molev [10], [12], [13], see also [19] and [18]. This base is called in the present paper the Gelfand-Tsetlin-Molev base. Mention that a basis of the same type in the case \mathfrak{sp}_{2n} was firstly constructed by Zhelobenko [14].

In the construction of the the Gelfand-Tsetlin-Molev base the key role is played by the Mickelsson-Zhelobenko algebra $Z(\mathfrak{o}_N, \mathfrak{o}_{N-2})$. (see it's definitions in sec. 5), which acts on the space of \mathfrak{o}_{N-2} -highest vectors of a \mathfrak{o}_N -representation. The elements of this algebra are called raising operators.

The first main result of the paper is the following. We find explicitly an element of $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$, which acts on the space of \mathfrak{o}_{2n-1} -highest vectors of a \mathfrak{o}_{2n+1} -representation in the same way as the pfaffian $PfF_{\hat{n}}$.

This element is proportional to \check{z}_{n0} which is one of the canonical generators of the Mickelsson-Zhelobenko algebra. A coefficient of proportionality belongs to $U(\mathfrak{h}_{\mathfrak{o}_{2n-1}})$.

One can interpret the first main result as follows. Each operator acting on the space of \mathfrak{o}_{2n-1} -highest vectors of a \mathfrak{o}_{2n+1} -representation can be continued to an operator, acting on the whole \mathfrak{o}_{2n+1} -representation. This continuation commutes with the action of \mathfrak{o}_{2n-1} . We find this continuation for the element $C\check{z}_{n0}$ of the Mickelsson-Zhelobenko, $C \in U(\mathfrak{h}_{\mathfrak{o}_{2n-1}})$. For other raising operator such continuation are not known in an explicit way as far as we know.

The second main result are explicit formulae for the action of $PfF_{\hat{n}}$ in the Gelfand-Tsetlin-Molev base (Theorem 2).

Let us give the main definitions and describe the structure of the paper.

Let $\Phi = (\Phi_{ij})$, $i, j = 1, \dots, 2n$ be a skew-symmetric $2n \times 2n$ -matrix, whose matrix entries belong to a noncommutative ring.

Definition 1. The noncommutative pfaffian of Φ is defined by the formulae

$$Pf\Phi = \frac{1}{n!2^n} \sum_{\sigma \in S_{2n}} (-1)^\sigma \Phi_{\sigma(1)\sigma(2)} \dots \Phi_{\sigma(2n-1)\sigma(2n)},$$

Here σ is a permutation of the set $\{1, \dots, 2n\}$.

In the paper a split realization of \mathfrak{o}_N is used. The algebra \mathfrak{o}_N is generated by matrices $F_{ij} = E_{ij} - E_{-j-i}$, here E_{ij} are matrix units. When N is odd the indices i, j belong to the set $\{-n, \dots, -1, 0, 1, \dots, n\}$, where $n = \frac{N-1}{2}$. When N is even the indices i, j belong to the set $\{-n, \dots, -1, 1, \dots, n\}$, where $n = \frac{N}{2}$. Shortly in both cases this set of indices is denoted in the paper as $\{-n, \dots, n\}$.

In the paper the following noncommutative pfaffians are considered.

Definition 2. Let F be the matrix $F = (F_{ij})$. For every subset $I \subset \{-n, \dots, n\}$ which consists of an even number k of elements define a submatrix F_I by the formulae $F_I = (F_{ij})_{-i, j \in I}$. Put

$$PfF_I = Pf(F_{-ij})_{-i, j \in I}.$$

In [18] the author in terms of these pfaffians defines some special elements of $U(\mathfrak{o}_N)$ called the Capelli elements. These elements are $C_k = \sum_{I \subset \{-n, \dots, n\}, |I|=k} PfF_I PfF_{-I}$, $k = 2, 4, \dots, [\frac{N}{2}]$. It is proved that the elements C_k belong to the center of $U(\mathfrak{o}_N)$.

Mention that the relation between commutative pfaffians were intensively studied in [1], [2], [3], [4]. Also some relation between noncommutative pfaffians PfF_I were derived in [5], [?].

The structure if the paper is the following.

In sec. 3.1 the action of pfaffians on weight vectors is investigated. It is proved that a weight vector is mapped to a weight vector, a rule according to which the weight changes is established (Proposition 1).

In sec. 2 some facts about the split realization of the orthogonal algebra are given. In sec. 3.2, it is proved that in the case $N = 2n + 1$ the pfaffians

$$PfF_{\frown_n} := PfF_{\{-n+1, \dots, n\}} \text{ and } PfF_{\hat{n}} := PfF_{\{-n, \dots, n-1\}}$$

commute with elements of the subalgebra $\mathfrak{o}_{2n-1} = \langle F_{ij} \rangle$, $-n+1 \leq i, j \leq n-1$ (Corrolary 3).

In particular in the case \mathfrak{o}_{2n+1} the pfaffian PfF_{\frown_n} diminishes the n -th component of the weight by one and $PfF_{\hat{n}}$ raises the n -th component of the weight by one (Corollary 1).

In sec 3.3 some formulaes involving pfaffians are proved.

In sec. 4 using two the facts mentioned above an important observation is done. The pfaffians PfF_{\frown_n} and $PfF_{\hat{n}}$ act on the space of \mathfrak{o}_{2n-1} -highest vectors of a \mathfrak{o}_{2n+1} -representation with a fixed \mathfrak{o}_{2n-1} -weight.

Then we find an element of the the Mickelsson-Zhelobenlo algebra that acts on the space of \mathfrak{o}_{2n-1} -highest vectors in the same way as the pfaffian $PfF_{\hat{n}}$.

There exists a projection from $U(\mathfrak{o}_N)$ to $Z(\mathfrak{o}_N, \mathfrak{o}_{N-2})$. The element of the algebra that acts in the same way as the pfaffian is an image of $PfF_{\hat{n}}$ under this projection. After some calculations in sec. 5.2, 5.3 we do it (Theorem 1).

Using this result in sec. 6 we find explicit formulae for the action of the pfaffian $PfF_{\hat{n}}$ on the base in the Gelfand-Tsetlin-Molev base (Theorem 2).

Application of these calculations are presented in Sec. ???. We formulate a problem of construction of an additional quantum number for classifications of quasi-spin states and establish its relation to the Gelfans-Tsetlin-Molev bases for \mathfrak{o}_5 -representations.

In subsection ?? Theorem ?? is proved, in which we construct an additional quantum number using pfaffians.

In appendix an action of a pfaffian in a tensor representation is investigated. The base vectors are encoded by orthogonal Young tableaux [21]. An action of

a pfaffian on tensor products of vectors of a standard representation is found (Propositions 6,7,8,9). Then theorem 4 is proved which gives an information about the action on base vectors given by Young tableaux. The image of a base vector in this theorem is expressed as a linear combination of not necessarily orthogonal Young tableaux. Thus Theorem 4 does not give explicit formulaes of the action of a pfaffian in the bases formed by orthogonal Young tableaux.

2 The split realization of the orthogonal algebra

The orthogonal algebra \mathfrak{o}_N is a tangent space at the unit to the group of linear transformations that preserve a nondegenerate quadratic form. Let G be a matrix of the form. A matrix f belongs to the algebra \mathfrak{o}_N if $f^t G + Gf = 0$. We use the following indexation of rows and columns of matrices f . When N is odd the indices i, j belong to the set $\{-n, \dots, -1, 0, 1, \dots, n\}$, where $n = \frac{N-1}{2}$. When N is even the indices i, j belong to the set $\{-n, \dots, -1, 1, \dots, n\}$, where $n = \frac{N}{2}$. Shortly this set of indices is denoted in the paper as $\{-n, \dots, n\}$.

In the paper the split realization of the algebra \mathfrak{o}_N is used. It corresponds to the following choice of the form: $G = (\delta_i^{-j})$.

In this realization the algebra \mathfrak{o}_N is generated by elements $F_{ij} = E_{ij} - E_{-j-i}$. The only linear relations between these elements are $F_{ij} = -F_{-j-i}$.

Commutation relations between these elements are

$$[F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj} - \delta_{-ki} F_{-jl} + \delta_{-lj} F_{k-i}.$$

One can prove that elements $F_{-n-n}, \dots, F_{-1-1}$ form a bases in the Cartan subalgebra, and elements $F_{ij}, j < -i$ are root elements.

The precise correspondence is the following. Let e_i be the element F_{ii}^* in the dual space to the Cartan subalgebra. Put $e_{-r} := -e_r$ and $e_0 = 0$. Then the element F_{ij} corresponds to the root $e_i - e_j$ (§13 in the chapter 8 in [15]).

3 Noncommutative pfaffians.

In this section some properties of noncommutative pfaffians are obtained. Firstly the action on weight vectors of a representation is investigated. Then commutators of pfaffians with elements F_{ij} are calculated. Finally some summation formulaes involving pfaffians are proved.

3.1 Action of a pfaffian on a weight vector.

Remind that e_i denotes the standard base vectors F_{ii}^* of dual space to the Cartan subalgebra.

Proposition 1. *Let V be a representation of \mathfrak{o}_N . Under the action of the pfaffian PfF_I a weight vector with the weight μ is mapped to a weight vector with the weight $\mu - \sum_{i \in I} e_i$.*

Proof. If v is a weight vector in a representation of \mathfrak{o}_N with the weight μ , g_α is a root element in \mathfrak{o}_N corresponding to the root α , then $g_\alpha v_\mu$ is a weight vector of to the weight $\alpha + \mu$.

Consider the vector $PfF_I v$. By definition one has

$$PfF_I = \frac{1}{\frac{k}{2}! 2^{\frac{k}{2}}} \sum_{\sigma \in S^k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \dots F_{-\sigma(i_{k-1})\sigma(i_k)}.$$

To prove the proposition it suffices to show that every summand changes the weight by subtracting of the same expression $-\sum_{i \in I} e_i$. Using the correspondence between roots and elements F_{ij} from the sec. 2 one gets the following. When one acts by $F_{-\sigma(i_1)\sigma(i_2)} \dots F_{-\sigma(i_{k-1})\sigma(i_k)}$ on v then to the weight the vector

$$e_{-\sigma(i_1)} - e_{\sigma(i_2)} - \dots + e_{-\sigma(i_{k-1})} - e_{\sigma(i_k)} = -\sum_{i \in I} e_i$$

is added. This proves the proposition. \square

Consider the most interesting case $\mathfrak{o}_N = \mathfrak{o}_{2n+1}$ and $|I| = 2n$.

Corrolary 1. *Let $\mathfrak{o}_N = \mathfrak{o}_{2n+1}$.*

The action of $PfF_{\widehat{-n}}$ adds the vector $-\sum_{i \in I} e_i = -e_n$ to the weight.

The action of $PfF_{\widehat{n}}$ adds the vector $-\sum_{i \in I} e_i = -e_{-n} = e_n$ to the weight.

3.2 Commutators of pffians and F_{ij} .

Lemma 1. *Let $I = \{i_1, \dots, i_k\}$, where k is even. Then the commutator $[PfF_I, F_{j_1-j_2}]$ is calculated according to the following rule.*

1. *If $j_1, j_2 \notin I$, then $[PfF_I, F_{j_1-j_2}] = 0$.*
2. *If $j_1 \in I, j_2 \notin I$, then $[PfF_I, F_{j_1-j_2}] = PfF_{I_{j_1 \rightarrow -j_2}}$.*
3. *If $j_1 \notin I, j_2 \in I$, then $[PfF_I, F_{j_1-j_2}] = -PfF_{I_{j_2 \rightarrow -j_1}}$.*
4. *If $j_1 \in I, j_2 \in I$, then $[PfF_I, F_{j_1-j_2}] = PfF_{I_{j_1 \rightarrow -j_2}} - PfF_{I_{j_2 \rightarrow -j_1}}$.*

Proof. One can identify E_{ij} with $e_i \otimes e_{-j}$. Then F_{ij} is identified with $e_i \wedge e_{-j}$. Remind that

$$PfF_I = \frac{1}{2^{\frac{k}{2}} \frac{k}{2}!} \sum_{\sigma \in S_k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \dots F_{-\sigma(i_{k-1})\sigma(i_k)}.$$

Thus PfF_I with indexing set $I = \{i_1, \dots, i_k\}$ is identified with the polyvector $e_{-i_1} \wedge \dots \wedge e_{-i_k}$.

This identification is compatible with the action of \mathfrak{o}_N . Thus

$$\begin{aligned} [PfF_I, F_{j_1-j_2}] &= -[F_{j_1-j_2}, PfF_I] = \\ &= F_{j_1-j_2} e_{i_1} \wedge e_{i_2} \dots \wedge e_{i_k} + e_{i_1} \wedge F_{j_1-j_2} e_{i_2} \dots \wedge e_{i_k} + e_{i_1} \wedge e_{i_2} \dots \wedge F_{j_1-j_2} e_{i_k}. \end{aligned}$$

One has $F_{j_1-j_2}e_{-i_1} = e_{j_2}$ if $j_2 = i_1$ and $F_{j_1-j_2}e_{-i_1} = -e_{j_2}$ if $j_1 = i_1$.

Suggest that $\{j_1, j_2\} \cap I = \emptyset$. Then $[PfF_I, F_{j_1-j_2}] = 0$.

Suggest that $j_1 \in I, j_2 \notin I$. Then $j_1 = i_t$ and the only nonzero summand is that containing $F_{j_1 j_2}e_{-i_t}$. Thus we have $[PfF_I, F_{j_1-j_2}] = -PfF_{I_{j_1 \mapsto -j_2}}$. Here $I_{j_1 \mapsto -j_2}$ is obtained from I by replacing the index j_1 to j_2 .

The case $j_1 \notin I, j_2 \in I$ is considered in the same manner.

Suggest that $\{j_1, j_2\} \subset I$. That is $j_1 = e_{i_t}, j_2 = e_{i_s}$. Then in the sum there are two nonzero summands one contains $F_{j_1 j_2}e_{i_t}$, the other contains $F_{j_1 j_2}e_{i_s}$. Each of them is a wedge product with a new indexing set. So one gets that $[PfF_I, F_{j_1 j_2}] = PfF_{I_{j_1 \mapsto -j_2}} - PfF_{I_{j_2 \mapsto -j_1}}$.

□

Write the formulae from Lemma 1 in the case $N = 2n + 1$ for pfaffians

$$PfF_{\hat{i}} := PfF_{\{-n, \dots, \hat{i}, \dots, n\}}.$$

Corrolary 2. *Let $N = 2n + 1$, then*

1. $[PfF_{\hat{i}}, F_{i-j}] = (-1)^{i+j} PfF_{\hat{j}}$.
2. $[PfF_{\hat{j}}, F_{i-j}] = -(-1)^{i+j} PfF_{\hat{i}}$.
3. If $k \neq i, j$ $[PfF_{\hat{k}}, F_{i-j}] = 0$.

Proof. Take formulae for $[PfF_I, F_{i-j}] = 0$, written in Lemma 1.

If $i \notin I$ or $j \notin I$, then the considered commutator is zero.

If $i, j \in I$, then the considered commutator equals to $PfF_{I_{i \mapsto -j}} - PfF_{I_{j \mapsto -i}}$.

Consider this case in details.

Note that I contains all indices except one.

If $-i, -j \in I$ (that is $PfF_I \neq PfF_{\hat{i}}, PfF_{\hat{j}}$), then both summands $PfF_{I_{i \mapsto -j}}, PfF_{I_{j \mapsto -i}}$ vanish.

If $-i \notin I$, then $PfF_{I_{i \mapsto -j}} = 0, PfF_{I_{j \mapsto -i}} = PfF_{\{-n, \dots, -i \text{ at the place } j, \dots, \hat{i}, \dots, n\}} = (-1)^{i+j+1} PfF_{\hat{j}}$. That is in this case $[PfF_{\hat{i}}, F_{i-j}] = (-1)^{i+j} PfF_{\hat{j}}$.

If $-j \notin I$, then $PfF_{I_{j \mapsto -i}} = 0, PfF_{I_{i \mapsto -j}} = PfF_{\{-n, \dots, -j \text{ at the place } i, \dots, \hat{j}, \dots, n\}} = (-1)^{i+j+1} PfF_{\hat{i}}$. That is in this case $[PfF_{\hat{j}}, F_{i-j}] = (-1)^{i+j+1} PfF_{\hat{i}}$.

□

Corrolary 3. *In the case \mathfrak{o}_{2n+1} the pfaffians $PfF_{\hat{n}}, PfF_{\hat{-n}}$ commute with elements $F_{ij}, -n < i, j < n$, that span the subalgebra \mathfrak{o}_{2n-1}*

3.3 Some formulas involving pfaffians.

In this subsection some summation formulae are proved.

Lemma 2. $PfF_I = \frac{(\frac{p}{2})!(\frac{q}{2})!}{(\frac{k}{2})!} \sum_{I=I' \sqcup I'', |I'|=p, |I''|=q} (-1)^{(I'I'')} PfF_{I'} PfF_{I''}$.

Here $(-1)^{(I'I'')}$ is a sign of a permutation of the set $I = \{i_1, \dots, i_k\}$ that places first the subset $I' \subset I$ and then the subset $I'' \subset I$.

The numbers p, q are even fixed numbers, they satisfy $p + q = k = |I|$.

Proof. By definition one has

$$PfF_I = \frac{1}{2^{\frac{k}{2}}(\frac{k}{2})!} \sum_{\sigma \in S_k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \dots F_{-\sigma(i_{k-1}),\sigma(i_k)}.$$

The summand $(-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \dots F_{-\sigma(i_{k-1}),\sigma(i_k)}$ can be written as

$$(-1)^{(I'I'')} (-1)^{\sigma'} F_{-\sigma'(i'_1)\sigma'(i'_2)} \dots F_{-\sigma'(i'_{p-1}),\sigma'(i'_p)} (-1)^{\sigma''} F_{-\sigma''(i''_1)\sigma''(i''_2)} \dots F_{-\sigma''(i''_{q-1}),\sigma''(i''_q)}.$$

Here $I' = \{i'_1, \dots, i'_p\}$ is the set of indices $\{\sigma(i_1), \dots, \sigma(i_p)\}$ placed in a natural order, $I'' = \{i''_1, \dots, i''_q\}$ is set of indices $\{\sigma(i_{p+1}), \dots, \sigma(i_k)\}$ placed in a natural order, σ' is a permutation $\{\sigma(i_1), \dots, \sigma(i_p)\}$ of the set I' and σ'' is a permutation of the set I'' defined in a similar way. Note that $(-1)^{(I'I'')} (-1)^{\sigma'} (-1)^{\sigma''} = (-1)^\sigma$.

The mapping $\sigma \mapsto I', I'', \sigma', \sigma''$ is bijective.

Thus the pfaffian can be written as

$$\begin{aligned} & \frac{(\frac{k}{2})!(\frac{k}{2})!}{(\frac{k}{2})!} \sum_{I=I' \sqcup I'', |I'|=p, |I''|=q} (-1)^{(I'I'')} \frac{1}{2^{\frac{k}{2}}(\frac{k}{2})! (\frac{k}{2})!} \sum_{\sigma'} (-1)^{\sigma'} (-1)^{\sigma''} F_{-\sigma'(i'_1)\sigma'(i'_2)} \dots \\ & \dots F_{-\sigma'(i'_{p-1}),\sigma'(i'_p)} F_{-\sigma''(i''_1)\sigma''(i''_2)} \dots F_{-\sigma''(i''_{q-1}),\sigma''(i''_q)} = \\ & = \frac{(\frac{k}{2})!(\frac{k}{2})!}{(\frac{k}{2})!} \sum_{I=I' \sqcup I'', |I'|=p, |I''|=q} (-1)^{(I'I'')} PfF_{I'} PfF_{I''} \quad \square \end{aligned}$$

Corrolary 4. If $|I| = k$, then

$$PfF_I = \frac{1}{\frac{k}{2}+1} \sum_{I=I' \sqcup I''} (-1)^{(I'I'')} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} PfF_{I'} PfF_{I''}.$$

Lemma 3. Let $-n \in I$. Then $PfF_I =$

$$= \sum_{i \in I \setminus \{-n\}} \sum_{I \setminus \{-n, i\} = I' \sqcup I''} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - ni I'')} PfF_{I'} F_{ni} PfF_{I''}.$$

Here $(-1)^{(I' - ni I'')}$ is a sign of the permutation $(I', -n, i, I'')$ of the set I .

Proof. By definition one has

$$PfF_I = \frac{1}{2^{\frac{k}{2}}(\frac{k}{2})!} \sum_{\sigma \in S_k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \dots F_{-\sigma(i_{k-1}),\sigma(i_k)}.$$

Since $F_{ij} = -F_{j-i}$ the summation can be taken only over such permutation such that $\sigma(i_{2t-1}) < \sigma(i_{2t})$. But if the summation is done in such a way the multiple $\frac{1}{2^{\frac{k}{2}}}$ must be omitted.

Fix a such a permutation σ and find a place such that $(\sigma(i_{2t-1}), \sigma(i_{2t})) = (-n, i)$. The summand $(-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \dots F_{-\sigma(i_{k-1}),\sigma(i_k)}$ can be written as

$$(-1)^{(I' in I'')} (-1)^{\sigma'} F_{-\sigma'(i'_1)\sigma'(i'_2)} \dots F_{-\sigma'(i'_{p-1}),\sigma'(i'_p)} F_{-ni} (-1)^{\sigma''} F_{-\sigma''(i''_1)\sigma''(i''_2)} \dots F_{-\sigma''(i''_{q-1}),\sigma''(i''_q)}.$$

Here $I' = \{i'_1, \dots, i'_p\}$ is the set of indices $\{\sigma(i_1), \dots, \sigma(i_{2t-2})\}$ placed in the natural order, $I'' = \{i''_1, \dots, i''_q\}$ is set of indices $\{\sigma(i_{2t+1}), \dots, \sigma(i_k)\}$ placed in the natural order, σ' is a permutation $\{\sigma(i_1), \dots, \sigma(i_{2t-2})\}$ of the set I' and σ'' is a permutation of the set I'' defined in a similar way. Note that $(-1)^{(I' - ni I'')} (-1)^{\sigma'} (-1)^{\sigma''} = (-1)^\sigma$. The permutation σ' satisfies the condition $\sigma'(i'_{2t-1}) < \sigma'(i'_{2t})$ as well as the permutation σ'' .

The mapping $\sigma \mapsto I', I'', \sigma', \sigma''$ is bijective (since $\sigma(i_{2t-1}) < \sigma(i_{2t})$).

Thus the pfaffian can be written as

$$\begin{aligned}
& \sum_{i \in I \setminus \{n\}} \sum_{I \setminus \{i, n\} = I' \sqcup I''} \frac{(\frac{|I'|}{2})! (\frac{|I''|}{2})!}{(\frac{k}{2})!} \frac{1}{(\frac{|I'|}{2})! (\frac{|I''|}{2})!} (-1)^{(I' - niI'')} (-1)^{\sigma'} F_{-\sigma'(i'_1)\sigma'(i'_2)} \dots \\
& \dots F_{-\sigma'(i'_{p-1})\sigma'(i'_p)} F_{ni} (-1)^{\sigma''} F_{-\sigma''(i''_1)\sigma''(i''_2)} \dots F_{-\sigma''(i''_{q-1})\sigma''(i''_q)} = \\
& = \sum_{i \in I \setminus \{n\}} \sum_{I \setminus \{i, n\} = I' \sqcup I''} \frac{(\frac{|I'|}{2})! (\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - niI'')} Pf F_{I'} F_{ni} Pf F_{I''}
\end{aligned}$$

□

Lemma 4. $\Delta Pf F_I = \sum_{I' \sqcup I'' = I} (-1)^{(I' I'')} Pf F_{I'} \otimes Pf F_{I''}$

Here $(-1)^{(I' I'')}$ is a sign of a permutation of the set $I = \{i_1, \dots, i_k\}$ that places first the subset $I' \subset I$ and then places the subset $I'' \subset I$.

Proof. By definition one has

$$Pf F_I = \frac{1}{2^{\frac{k}{2}} (\frac{k}{2})!} \sum_{\sigma \in S_k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \dots F_{-\sigma(i_{k-1})\sigma(i_k)}.$$

Apply the comultiplication, one gets

$$\Delta Pf F_I = \frac{1}{2^{\frac{k}{2}} (\frac{k}{2})!} \sum_{\sigma \in S_k} (-1)^\sigma (F_{-\sigma(i_1)\sigma(i_2)} \otimes 1 + 1 \otimes F_{-\sigma(i_1)\sigma(i_2)}) \dots (F_{-\sigma(i_{k-1})\sigma(i_k)} \otimes 1 + 1 \otimes F_{-\sigma(i_{k-1})\sigma(i_k)})$$

The product

$$(F_{-\sigma(i_1)\sigma(i_2)} \otimes 1 + 1 \otimes F_{-\sigma(i_1)\sigma(i_2)}) \dots (F_{-\sigma(i_{k-1})\sigma(i_k)} \otimes 1 + 1 \otimes F_{-\sigma(i_{k-1})\sigma(i_k)})$$

equals to

$$\sum_{I = J' \sqcup J''} F_{-\sigma(j'_1)\sigma(j'_2)} \dots F_{-\sigma(j'_{p-1})\sigma(j'_p)} \otimes F_{-\sigma(j''_1)\sigma(j''_2)} \dots F_{-\sigma(j''_{q-1})\sigma(j''_q)}.$$

Here $J' = \{j'_1, j'_2, \dots, j'_p\}$, $J'' = \{j''_1, j''_2, \dots, j''_q\}$ are subset of I , such that $I = J' \sqcup J''$ and also the following condition are satisfied. If $\sigma(i_{2t-1}) \in J'$ then $\sigma(i_{2t}) \in J'$, if $\sigma(i_{2t-1}) \in J''$ then $\sigma(i_{2t}) \in J''$. In other word the partitions $I = J' \sqcup J''$ must induce a division of $\frac{k}{2}$ pairs $(\sigma(i_{2t-1}), \sigma(i_{2t}))$

The summand

$$(-1)^\sigma F_{-\sigma(j'_1)\sigma(j'_2)} \dots F_{-\sigma(j'_{p-1})\sigma(j'_p)} \otimes F_{-\sigma(j''_1)\sigma(j''_2)} \dots F_{-\sigma(j''_{q-1})\sigma(j''_q)}$$

indexed by $\sigma \in \text{Aut}(I)$, J', J'' can be written as the following expression

$$(-1)^{(I' I'')} (-1)^{\sigma'} F_{-\sigma'(i'_1)\sigma'(i'_2)} \dots F_{-\sigma'(i'_{p-1})\sigma'(i'_p)} \otimes (-1)^{\sigma''} F_{-\sigma''(i''_1)\sigma''(i''_2)} \dots F_{-\sigma''(i''_{q-1})\sigma''(i''_q)}$$

indexed by $I', I'', \sigma' \in \text{Aut}(I'), \sigma'' \in \text{Aut}(I'')$. Here I' is the set $\{\sigma(j'_1), \dots, \sigma(j'_p)\}$ written in the natural order, I'' is the set $\{\sigma(j''_1), \dots, \sigma(j''_q)\}$ written in a natural order. The permutation σ' is the permutation $\{\sigma(j'_1), \dots, \sigma(j'_p)\}$ of J' and σ'' is a permutation of J'' defined in a similar way.

But the mapping $\sigma, J', J'' \mapsto I', I'', \sigma', \sigma''$ is not injective. To get the triple σ, J', J'' with the prescribed image $I', I'', \sigma', \sigma''$ one must divide $\frac{k}{2}$ pairs $\{(i_1, i_2), \dots, (i_{k-1}, i_k)\}$ into two subsets J' and J'' with $\frac{|I'|}{2}$ and $\frac{|I''|}{2}$ elements

respectively. Take a permutation σ , such that $\sigma(J') = \sigma'(I')$ (as order sets), and $\sigma(J'') = \sigma''(I'')$ (as order sets). The only freedom is the choice of two subsets J' and J'' . Thus the number of elements in the preimage equals to the number divisions of $\frac{k}{2}$ pairs into two subsets: one consists of $\frac{|I'|}{2}$ pairs and the other consists of $\frac{|I''|}{2}$ pairs. The number is $\frac{(\frac{k}{2})!}{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}$.

Thus ΔPfF_I can be written as

$$\frac{1}{2^{\frac{k}{2}}(\frac{k}{2})!} \sum_{I=I' \sqcup I''} \frac{(\frac{k}{2})!}{(\frac{|I'|}{2})!(\frac{|I''|}{2})!} (-1)^{(I' I'')} (\sum_{\sigma' \in \text{Aut}(I')} (-1)^{\sigma'} F_{-\sigma'(i'_1)\sigma'(i'_2)\dots\sigma'(i'_{p-1})\sigma'(i'_p)}) \otimes$$

$$\otimes (\sum_{\sigma'' \in \text{Aut}(I'')} (-1)^{\sigma''} F_{-\sigma''(i''_1)\sigma''(i''_2)\dots\sigma''(i''_{p-1})\sigma''(i''_p)}).$$

This expression equals $\sum_{I' \sqcup I'' = I} (-1)^{(I' I'')} PfF_{I'} \otimes PfF_{I''}$.

□

4 Pfaffians are raising operators

According to the sections 3.2 and 3.1 the following holds

1. The action of $PfF_{\widehat{n}}$ and $PfF_{\widehat{-n}}$ commutes with the action of the subalgebra \mathfrak{o}_{2n-1} , spanned by F_{ij} , $-n < i, j < n$.
2. Under the action of $PfF_{\widehat{-n}}$ a weight vector is mapped to a weight vector, the n -th component of the weight is diminished by 1. Under the action of $PfF_{\widehat{n}}$ a weight vector is also mapped to a weight vector, the n -th component of the weight is raised by 1.

The following lemma is proved.

Lemma 5. *The pfaffians $PfF_{\widehat{n}}$ and $PfF_{\widehat{-n}}$ act on the space of \mathfrak{o}_{2n-1} -highest vectors of a \mathfrak{o}_{2n+1} -representation. The \mathfrak{o}_{2n-1} -weight under this action is conserved.*

5 The Mickelson-Zhelobenko algebra and pfaffians.

There exists the Mickelsson-Zhelobenko algebra which as $PfF_{\widehat{n}}$ and $PfF_{\widehat{-n}}$ acts on the space of \mathfrak{o}_{2n-1} highest vectors of a \mathfrak{o}_{2n+1} representation. In the present section we find an element of this algebra which acts as $PfF_{\widehat{n}}$.

At first in the subsection 5.1 we give the definition of the Mickelsson-Zhelobenko algebra. There exists a mapping from $U(\mathfrak{o}_N)$ to the Mickelson-Zhelobenko algebra $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$. The image of the pfaffian $PfF_{\widehat{n}}$ is exactly an element of $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$, which acts on the space of \mathfrak{o}_{2n-1} -highest vectors as the pfaffian.

In subsections 5.2, 5.3 the images of some special pfaffian in $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ are found. Using this calculations at the end of the subsection 5.3 the image of $PfF_{\widehat{n}}$ in the Mickelsson-Zhelobenko algebra is found.

5.1 The Mickelson-Zhelobenko algebra

The Gelfand-Tsetlin-Molev's approach to a construction of a bases of a \mathfrak{o}_{2n+1} -representation is based on restrictions $\mathfrak{o}_{2n+1} \downarrow \mathfrak{o}_{2n-1}$, in contrast to the classical Gelfand-Tsetlin's approach which is based on restrictions $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$. The subalgebra $\mathfrak{o}_{2n-1} \subset \mathfrak{o}_{2n+1}$ is spanned by the elements F_{ij} , $-n < i, j < n$. The Cartan subalgebra $\mathfrak{h}_{\mathfrak{o}_{2n-1}}$ is a subalgebra in $\mathfrak{h}_{\mathfrak{o}_{2n+1}}$ and root vectors in \mathfrak{o}_{2n-1} are also root vectors in \mathfrak{o}_{2n+1} .

Remind a scheme of construction of a \mathfrak{o}_{2n+1} -representation V . An irreducible representation V of the algebra \mathfrak{o}_{2n+1} becomes reducible as a representation of \mathfrak{o}_{2n-1} . According to the scheme of Gelfand and Tsetlin in order to construct a base one firstly must know possible highest weights μ of irreducible \mathfrak{o}_{2n-1} -representations into which splits V . Secondly if a weight has multiplicity one must be able to construct a bases in the multiplicity space, that is in the space of \mathfrak{o}_{2n-1} -highest vectors with a fixed \mathfrak{o}_{2n-1} -weight μ .

Introduce a notation for this space.

Definition 3. Let V_μ^+ be a space of \mathfrak{o}_{2n-1} -highest vectors with the \mathfrak{o}_{2n-1} -weight μ in a \mathfrak{o}_{2n+1} -representation V .

To construct a base in V_μ^+ Molev used the Mickelson-Zhelobenko algebra acting on the space $\bigoplus_\mu V_\mu^+$.

Let us give a definition of this algebra, see also [17],[19], and the chapter 9 in [18].

Let \mathfrak{g} be a Lie algebra and let \mathfrak{k} be it's reductive subalgebra. The main example is $\mathfrak{g} = \mathfrak{o}_{2n+1}$ and $\mathfrak{k} = \mathfrak{o}_{2n-1}$. Let $\mathfrak{k} = \mathfrak{k}^- + \mathfrak{h} + \mathfrak{k}^+$ be a triangular decomposition. Let $R(\mathfrak{h})$ be a field of fractions of the algebra $U(\mathfrak{h})$. Denote as

$$U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} R(\mathfrak{h}).$$

Let

$$J' = U'(\mathfrak{g})\mathfrak{k}^+$$

be the left ideal in $U'(\mathfrak{g})$, generated by \mathfrak{k}^+ . Put

$$M(\mathfrak{g}, \mathfrak{k}) = U'(\mathfrak{g})/J'.$$

For every positive root α of the algebra \mathfrak{k} define

$$p_\alpha = 1 + \sum_{k=1}^{\infty} e_{-\alpha}^k e_\alpha^k \frac{(-1)^k}{k!(h_\alpha + \rho(h_\alpha) + 1) \dots (h_\alpha + \rho(h_\alpha) + k)},$$

here e_α is a root vector \mathfrak{k} , corresponding to α , h_α is a corresponding Cartan element, ρ is a half-sum of positive roots of \mathfrak{k} .

An order is normal if the following holds. Let a root be a sum of two roots, then it lies between them. Chose a normal ordering $\alpha_1 < \dots < \alpha_m$ of positive roots of \mathfrak{k} .

Put

$$p = p_{\alpha_1} \dots p_{\alpha_m}.$$

This element is called the extremal projector. It can be proved that nevertheless p is an infinite series its action on $M(\mathfrak{g}, \mathfrak{k})$ by left multiplication is well defined [17].

The following equalities hold: $e_\alpha p = p e_{-\alpha} = 0$, here α is a positive root of \mathfrak{k} .

Put

$$Z(\mathfrak{g}, \mathfrak{k}) = pM(\mathfrak{g}, \mathfrak{k}).$$

This is the Mickelson-Zhelobenko algebra. The multiplication in $Z(\mathfrak{g}, \mathfrak{k})$ is defined using the isomorphism $Z(\mathfrak{g}, \mathfrak{k}) = \text{Norm}J'/J'$, where $\text{Norm}J' = \{u \in U'(\mathfrak{g}) : J'u \subset J'\}$. Thus $Z(\mathfrak{g}, \mathfrak{k})$ is an associative algebra and a bimodule over $R(\mathfrak{h})$ [17].

Choose linear independent elements $v_1, \dots, v_n \in \mathfrak{g}$, such that $\langle v_1, \dots, v_n \rangle \oplus \mathfrak{k} = \mathfrak{g}$ as linear spaces over \mathbb{C} . Put $z_i = p v_i \text{mod} J'$. It can be proved that monomials $\check{z}_1^{m_1} \dots \check{z}_n^{m_n}$, $m_i \in \mathbb{Z}^+$, form a bases of $Z(\mathfrak{g}, \mathfrak{k})$ over $R(\mathfrak{h})$.

In the case $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ put

$$\check{z}_{i\pm n} = p F_{i,\pm n} \text{mod} J', \quad i = -n, \dots, n.$$

Notations are taken from [18]. There exists an obvious symmetry $\check{z}_{ij} = \check{z}_{-j-i}$. From previous considerations it follows that $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ is generated by elements \check{z}_{ia} , $i = 0, \dots, n$, $a = \pm n$ or \check{z}_{ai} , $i = 0, \dots, n$, $a = \pm n$.

Sometimes it is more useful to use the generators

$$z_{i\pm n} = \check{z}_{i\pm n} (f_i - f_{i-1}) \dots (f_i - f_{-n+1}),$$

where

$$f_i = F_{ii} + \rho_i, \text{ for } i > 0, \quad f_0 = -\frac{1}{2}, \quad f_{-i} = -f_i,$$

and

$$\rho_i = i - \frac{1}{2} \text{ for } i > 0 \text{ and } \rho_{-i} = -\rho_i.$$

In particular

$$z_{0n} = \check{z}_{0n} \prod_{i=1}^{n-1} (F_{ii} + i - \frac{1}{2}).$$

The Mickelsson-Zhelobenko algebra $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ acts on the space $\oplus_\mu V_\mu^+$ (see [18]). A weight μ changes under this action according to the following rule. Let a be $\pm n$ and $\mu + \delta_i = (\mu_1, \dots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_{n-1})$. Then for $i = 1, \dots, n-1$ the following holds

$$z_{ia} : V_\mu^+ \rightarrow V_{\mu+\delta_i}^+, \quad z_{ai} : V_\mu^+ \rightarrow V_{\mu-\delta_i}^+$$

Elements z_{0a} do not change a \mathfrak{o}_{2n-1} -weight, that is they map V_μ^+ into itself.

The pfaffians $PfF_{\hat{n}}$, $PfF_{\hat{-n}}$, as it was pointed out in the lemma 5, also act on each space V_μ^+ .

Obviously the images $pPfF_{\hat{n}} \text{mod} J'$, $pPfF_{\hat{-n}} \text{mod} J'$ of pfaffians in the Mickelsson-Zhelobenko algebra act in the same way as the corresponding pfaffians. In the next section it is proved that $pPfF_{\hat{n}} \text{mod} J' = C \check{z}_{n0}$, where $C \in U(\mathfrak{h}_{\mathfrak{o}_{2n-1}})$. The element C is calculated explicitly.

5.2 Images of pfaffians in the Mickelson-Zhelobenko algebra I

Definition 4. A product of root and Cartan elements in the universal enveloping algebra is called normally ordered if in it at first (from the left side) the negative root elements occur, then Cartan elements occur and at the end positive root elements occur.

Every product of root and Cartan elements equals to a sum of normally ordered products.

Proposition 2. Let $I \subset \{-n+1, \dots, n-1\}$ be a subset which is not symmetric with respect to zero. Then $pPfF_I = 0$ in $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ or in $Z(\mathfrak{o}_{2n}, \mathfrak{o}_{2n-2})$.

Proof. According to the definition a pfaffian a sum over permutations. The summands are products of root vectors and Cartan elements of \mathfrak{o}_N

The sum of root corresponding to element of each product equals $-\sum_{i \in I} e_i$. Since the set I is nonsymmetric one has $-\sum_{i \in I} e_i \neq 0$.

Impose a normal ordering in every summand. When one does the normal ordering new summands appear. But from the equality $[e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha+\beta}$ it follows that the sum of roots corresponding to the elements of these new products is again $-\sum_{i \in I} e_i$.

Since $-\sum_{i \in I} e_i \neq 0$ in every normally ordered summand in the pfaffian there is a root element. These elements either are zero modulo J' , if there is a positive root element, or become vanish after multiplication by p , if there is a negative root element.

□

Let us give formula for the image of a pfaffian whose indexing set I is symmetric and is contained in $\{-n+1, \dots, n-1\}$. In this case the calculation of the image in the Mickelsson-Zhelobenko algebra is equivalent to the calculation of the image of the pfaffian under the Harish-Candra homomorphism. This calculation was done in [20] (proposition 7.1), the result is the following.

Proposition 3. [20] $PfF_I = \frac{1}{(\frac{k}{2})!} D_{\frac{k}{2}}(F_{i_1 i_1}, \dots, F_{i_{\frac{k}{2}} i_{\frac{k}{2}}})$,
where $D_r(h_1, \dots, h_r) = \prod_{i=1}^r (h_i - \frac{r}{2} + i)$

5.3 Images of pfaffians in the Mickelsson-Zhelobenko algebra II

In the previous subsection the images in $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ or $Z(\mathfrak{o}_{2n}, \mathfrak{o}_{2n-2})$ of pfaffians PfF_I were found, where $I \subset \{-n+1, \dots, \widehat{0}, \dots, n-1\}$.

Now let us found an image in $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ of the pfaffian $PfF_{\widehat{n}}$.

To formulate the next theorem define a polynomial C_n .

Definition 5. Let $C_{n-1}(h_1, \dots, h_{n-1}) = (-1)^{n-1} D_{n-1}(h_1, \dots, h_{n-1}) - 4 \sum_{i=1}^{n-1} (-1)^{t+1} D_{n-2}(h_1, \dots, \widehat{h_i}, \dots, h_{n-1})$

Theorem 1. *The image of $PfF_{\tilde{n}}$ in $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ equals $\tilde{z}_{n0}C_{n-1}(F_{11}, \dots, F_{(n-1)(n-1)})$.*

Proof. Take a set of indices of type $I = \{-n, -i\frac{k-2}{2}, \dots, -i_1, 0, i_1, \dots, i\frac{k-2}{2}\}$

By Lemma 3 the following equality takes place

$$PfF_I = \sum_{i \in I \setminus \{-n\}} \sum_{I' \sqcup I'' = I \setminus \{i, -n\}} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - niI'')} PfF_{I'} F_{ni} PfF_{I''}.$$

To find the image in the Mickelson-Zhelobenko algebra of the sum $\sum_{I' \sqcup I'' = I \setminus \{i, -n\}}$ divide the summands into three groups: 1) those for which $i = 0$, 2) those for which $i < 0$, 3) those for which $i > 0$.

Let us find the image of summands for which $i = 0$. In this case $PfF_{I'}$ and $PfF_{I''}$ commute with F_{n0} . Note that $(-1)^{(I' - n0I'')} = (-1)^{(I' I'')} (-1)^{\frac{k}{2}-1}$ (to prove this firstly move $-n, 0$ to two first places and then move 0 to the right place, then the signs $(-1)^{|I'|}$, $(-1)^{|I''|-1}$, $(-1)^{\frac{k}{2}}$ appear).

Using the corollary 4 one gets that the image sum these summands equals

$$\begin{aligned} (-1)^{\frac{k}{2}-1} \sum_{I' \sqcup I'' = I \setminus \{-n, 0\}} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' I'')} PfF_{I'} PfF_{I''} F_{n0} = \\ \frac{1}{\frac{k}{2}} (-1)^{\frac{k}{2}-1} PfF_{I \setminus \{-n, 0\}} F_{n0} = (-1)^{\frac{k}{2}-1} PfF_{I \setminus \{-n, 0\}} F_{n0}. \end{aligned}$$

Since the sets of indices $\pm(I \setminus \{-n, 0\})$ and $\{-n, 0\}$ do not intersect, one can apply the projector p and equivalence $mod J'$ to each multiple.

Thus the image of these summands is

$$\tilde{z}_{n0}(pPfF_{I \setminus \{-n, 0\}} mod J').$$

Found the image of summands

$$\frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - niI'')} PfF_{I'} F_{ni} PfF_{I''},$$

for which $i \neq 0$. Let $i > 0$. Then change F_{ni} and $PfF_{I''}$. One obtains an expression

$$\frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - niI'')} (PfF_{I'} PfF_{I''} F_{ni} - PfF_{I'} [PfF_{I''}, F_{ni}]).$$

Now let $i < 0$. Change F_{ni} and $PfF_{I'}$, one gets

$$\frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - niI'')} (F_{ni} PfF_{I'} PfF_{I''} - [PfF_{I'}, F_{ni}] PfF_{I''}).$$

Consider the case $i > 0$. In the last expression the first summand has a zero image in the Mickelson-Zhelobenko algebra by the following reason. The sum of roots corresponding to the elements F_{ij} that participate in the expression for $PfF_{I'} F_{ni} PfF_{I''}$ equals to e_n . The element F_{ni} corresponds to the root $e_n - e_i$.

Thus the sum of roots corresponding to the elements $PfF_{I'}PfF_{I''}$ equals $-e_i$. Express $PfF_{I'}PfF_{I''}$ as a sum of normally ordered products. Since $i > 0$ than in every obtained normally product there is a negative root element of the algebra \mathfrak{o}_{2n-1} . Thus after applying the extremal projector p the expression $PfF_{I'}PfF_{I''}$ vanishes.

In the case $i < 0$ it is similarly proved that the first summand has a zero image in the Mickelsson-Zhelobenko algebra.

Now consider the second summand

$$-PfF_{I'}[PfF_{I''}, F_{ni}]$$

in the case $i > 0$ or

$$-[PfF_{I'}, F_{ni}]PfF_{I''}$$

in the case $i < 0$. In the first case if $-i \notin I''$ it is zero and it equals to $-PfF_{I'}PfF_{I''} \mid_{-i \mapsto -n}$ otherwise. In the second case if $-i \notin I'$ it is zero and it equals to $-PfF_{I'} \mid_{-i \mapsto -n} PfF_{I''}$ otherwise.

Thus the image of summands for which $i \neq 0$ equals to the image of the expression

$$\begin{aligned} & - \sum_{i \in I \setminus \{-n\}, i > 0} \sum_{I' \sqcup I'' = I \setminus \{-n, i\}, -i \in I''} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - niI'')} PfF_{I'} PfF_{I''} \mid_{-i \mapsto -n} \\ & - \sum_{i \in I \setminus \{-n\}, i < 0} \sum_{I' \sqcup I'' = I \setminus \{-n, i\}, -i \in I'} \frac{(\frac{|I'|}{2})!(\frac{|I''|}{2})!}{(\frac{k}{2})!} (-1)^{(I' - niI'')} PfF_{I'} \mid_{-i \mapsto -n} PfF_{I''} \end{aligned}$$

Let us prove a proposition.

Proposition 4. *The expression above equals*

$$-2 \sum_{t = -\frac{k}{2}, \neq 0}^{\frac{k}{2}} (-1)^{\frac{k}{2} - t - 1} \sum_{J' \sqcup J'' = I \setminus \{\pm i\}} \frac{(\frac{|J'|}{2})!(\frac{|J''|}{2})!}{(\frac{k}{2})!} (-1)^{(J'J'')} PfF_{J'} PfF_{J''}. \quad (1)$$

Proof. To prove this let us firstly calculate the sign $(-1)^{(I' - niI'')}$. The sign $(-1)^{(I' - niI'')}$ differs from the sign $(-1)^{(I'I'')}$ by the sign of the permutation which moves $-n, i$ to their right places. This permutation can be done as follows: first of all move $-n, i$ to two last places, then move i to it's right place. If $i = i_t$, then

$$(-1)^{(I' - niI'')} = (-1)^{(I'I'')} (-1)^{(|I'| + |I''| + \frac{k-2}{2} - t)} = (-1)^{\frac{k}{2} - t - 1} (-1)^{(I'I'')}.$$

Secondly compare $PfF_{I'} \mid_{-i \mapsto -n}$, $PfF_{I''} \mid_{-i \mapsto -n}$ and $PfF_{(I' \setminus \{-i\}) \cup \{-n\}}$, $PfF_{(I'' \setminus \{-i\}) \cup \{-n\}}$ respectively. Here it is assumed that $i \in I'$ and $i \in I''$. In all these expressions at first, the index $-i$ is changed to n , but then in the last two expressions the new set of indices is naturally ordered. Thus $PfF_{I'} \mid_{-i \mapsto -n}$ and $PfF_{(I' \setminus \{-i\}) \cup \{-n\}}$, $PfF_{I''} \mid_{-i \mapsto -n}$ and $PfF_{(I'' \setminus \{-i\}) \cup \{-n\}}$, differ by the sign of this ordering.

For summands in the sum $\sum_{i \in I \setminus \{-n\}, i < 0} \sum_{I' \sqcup I'' = I \setminus \{-ni\}, -i \in I''}$ denote

$$J' := (I' \setminus \{-i\}) \cup \{-n\}, \quad J'' := I''.$$

One obtains

$$(-1)^{(I' I'')} Pf F_{I'} \mid_{-i \mapsto -n} Pf F_{I''} = (-1)^{(J' J'')} Pf F_{J'} Pf F_{J''}.$$

The sign that appears after the ordering is contained in $(-1)^{(J' J'')}$.

Analogously for the summands in the sum $\sum_{i \in I \setminus \{-n\}, i < 0} \sum_{I' \sqcup I'' = I \setminus \{-ni\}, -i \in I''}$, denote

$$J' := I', \quad J'' := (I'' \setminus \{-i\}) \cup \{-n\}.$$

One obtains that

$$(-1)^{(I' I'')} Pf F_{I'} \mid_{-i \mapsto -n} Pf F_{I''} = (-1)^{(J' J'')} Pf F_{J'} Pf F_{J''}.$$

In both cases one has $J' \sqcup J'' = I \setminus \{\pm i\}$. Also $|J'| = |I'|$, $|I''| = |J''|$.

Note that a pair of sets J', J'' occurs twice. First as $(I' \setminus \{-i\}) \cup \{-n\}$, I'' , second as I' , $(I'' \setminus \{-i\}) \cup \{-n\}$.

Thus one obtains that the considered sum of images of summands for which $i \neq 0$ is given by the expression 1

$$2 \sum_{t=-\frac{k}{2}, \neq 0}^{\frac{k}{2}} (-1)^{\frac{k}{2}-t-1} \sum_{J' \sqcup J'' = I \setminus \{\pm i_t\}} \frac{(\frac{|J'|}{2})! (\frac{|J''|}{2})!}{(\frac{k}{2})!} (-1)^{(J' J'')} Pf F_{J'} Pf F_{J''}$$

The proposition is proved. \square

This expression 1 equals

$$2 \sum_{t=-\frac{k}{2}, \neq 0}^{\frac{k}{2}} (-1)^{\frac{k}{2}-t-1} Pf F_{I \setminus \{\pm i_t\}} = 4 \sum_{t=1}^{\frac{k}{2}} (-1)^{\frac{k}{2}-t-1} Pf F_{I \setminus \{\pm i_t\}}$$

(Corollary 4).

Finally one has

$$Pf F_I = (-1)^{\frac{k}{2}-1} \tilde{z}_{n0} (p Pf F_{I \setminus \{-n, 0\}} \text{mod} J') - 4 \sum_{t=1}^{\frac{k}{2}} (-1)^{\frac{k}{2}-t-1} Pf F_{I \setminus \{\pm i_t\}} \quad (2)$$

Note that $Pf F_{I \setminus \{-n, \pm i_t\}}$ is a pfaffian $Pf F_{I^t}$ for a new indexing set $I^t = I \setminus \{\pm i_t\}$.

This set is of the same type as I . Apply to each pfaffian $Pf F_{I^t}$ the equality (2). For each t there appears a summand

$$(-1)^{\frac{k-2}{2}-1} \tilde{z}_{n0} p Pf F_{I^t \setminus \{-n, 0\}} = (-1)^{\frac{k}{2}-2} \tilde{z}_{n0} p Pf F_{I \setminus \{\pm i, 0, -n\}}.$$

Also there appear summands

$$Pf F_{I^t \setminus \{\pm i_s\}} = \pm Pf F_{I \setminus \{\pm i_t, \pm i_s\}}.$$

But the sum of these summands over t and s is zero. Let $0 < t < s$. If this summand comes from the summand $PfF_{I \setminus \{\pm i_s\}}$ in (2), then it appears with the sign $(-1)^{\frac{k}{2}-s-1}(-1)^{(\frac{k}{2}-1)-t-1}$. If it comes from the summand $PfF_{I \setminus \{\pm i_t\}}$ in (2) then it has the sign $(-1)^{\frac{k}{2}-t-1}(-1)^{(\frac{k}{2}-1)-(s-1)-1}$. The sum of these signs is zero.

Hence

$$PfF_I = (-1)^{\frac{k}{2}-1} z_{n0} pPfF_{I \setminus \{-n, 0\}} - 4 \sum_{t=1}^{\frac{k}{2}} (-1)^{\frac{k}{2}-t-1} (-1)^{\frac{k}{2}-2} z_{n0} pPfF_{I \setminus \{\pm i, 0, -n\}}.$$

Apply the obtained formulae to $I = \hat{n}$. Recall that according to Proposition 3 one has $pPfF_{\widehat{0, \pm n}} = D_{n-1}(F_{11}, \dots, F_{(n-1)(n-1)})$, and $pPfF_{\widehat{\pm i, 0, \pm n}} = D_{n-2}(F_{11}, \dots, \widehat{F_{ii}}, \dots, F_{(n-1)(n-1)})$. Thus one proves Theorem. \square

6 Action of pfaffians in the multiplicity space and on the Gelfand-Tsetlin-Molev base.

Using Theorem 1 obtain formulae of the action of $PfF_{\hat{n}}$ on a base in V_{μ}^+ and then on the Gelfand-Tsetlin-Molev bases in V .

Everywhere below indices a, b belong to the set $\{-n, n\}$. We have introduced the notation $\rho_i = i - \frac{1}{2}$ for $i > 0$, and also $\rho_{-i} = -\rho_i$. Also we have denoted $f_i = F_{ii} + \rho_i$ for $i > 0$, $f_0 = \frac{1}{2}$ and $f_{-i} = -f_i$. Introduce a new notation

$$g_i = f_i + \frac{1}{2} \text{ for all } i.$$

Define elements $Z_{ab}(u)$ of the Mickelson-Zhelobenko algebra by the formulae (see §9.3 in [18])

$$Z_{ab}(u) = -(\delta_{ab}(u + \rho_n + \frac{1}{2}) + F_{ab}) \Pi_{i=-n+1}^{n-1}(u + g_i) + \sum_{i=-n+1}^{n-1} z_{ai} z_{ib} \Pi_{j=-n+1, j \neq i}^{n-1} \frac{u + g_i}{g_i - g_j}.$$

The mapping $s_{ab}(u) \mapsto u^{-2n} Z_{ab}(u)$ defines a homomorphism of the twisted yangian $Y(\mathfrak{o}_2) \rightarrow Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ (see §9.3 in [18]).

Since the Mickelson-Zhelobenko algebra $Z(\mathfrak{o}_{2n+1}, \mathfrak{o}_{2n-1})$ acts on $\oplus V_{\mu}^+$, the space $\oplus_{\mu} V_{\mu}^+$ becomes a $Y(\mathfrak{o}_2)$ -representation. It can be easily proved that the defined action of the yangian preserves the \mathfrak{o}_{2n-1} -weights and hence each space V_{μ}^+ is a $Y(\mathfrak{o}_2)$ -representation. This representation is a sum of two irreducible U, U' . The types of U, U' are known.

Let the highest weight of V be $\lambda = (\lambda_1, \dots, \lambda_n)$, where

$$0 \geq \lambda_1 \geq \dots \geq \lambda_n.$$

A base of the $Y(\mathfrak{o}_2)$ -module V_{μ}^+ explicitly is constructed as follows (all facts and notations are taken from §9.6 in [18]).

One takes a collection of numbers $(\sigma, \nu_1, \dots, \nu_n)$, satisfying the following conditions

$$0 \geq \nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \dots \geq \lambda_{n-1} \geq \nu_n \geq \lambda_n \quad (3)$$

$$0 \geq \nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \nu_n \quad (4)$$

The numbers ν_i are integers if λ_i are integers, and ν_i are half integers, if λ_i are half integers. The number σ equals 0 or 1, if $\nu_1 \neq 0$, and σ equals 0 if $\nu_1 = 0$.

Put

$$\gamma_i = \nu_i + \rho_i + \frac{1}{2}.$$

Let ξ be a highest weight vector of the \mathfrak{o}_{2n+1} -module V . There exists a base of the $Y(\mathfrak{o}_2)$ -module V_μ^+ formed by vectors

$$\xi_{\sigma, \nu} = z_{n0}^\sigma \prod_{i=1}^{n-1} z_{ni}^{\nu_i - \mu_i} z_{i-n}^{\nu_i - \lambda_i} \prod_{k=l_n}^{\gamma_n - 1} Z_{n-n}(k) \xi,$$

where

$$l_n = \lambda_n + \rho_n + \frac{1}{2}.$$

Put $\bar{\sigma} = \sigma + 1 \pmod{2}$

Write an action of z_{n0} on these vectors following §9.6 in [18].

If $\sigma = 0$, then

$$z_{n0} \xi_{\sigma, \nu} = \xi_{\bar{\sigma}, \nu}.$$

If $\sigma = 1$, then

$$z_{n0} \xi_{\sigma, \nu} = (-1)^n \sum_{j=1}^n \prod_{t=1, t \neq j}^n \frac{-\gamma_t^2}{\gamma_j^2 - \gamma_t^2} \xi_{\bar{\sigma}, \nu + \delta_j}.$$

From here one immediately obtains formulae of the action of the pfaffian $PfF_{\hat{n}}$ on V_μ^+ . These formulae are corollaries of Theorem 1 and the relation between z_{0n} and \check{z}_{0n} .

Lemma 6. *If $\sigma = 0$, then*

$$PfF_{\hat{n}} \xi_{\sigma, \nu} = C \xi_{\bar{\sigma}, \nu}.$$

If $\sigma = 1$, then

$$PfF_{\hat{n}} \xi_{\sigma, \nu} = (-1)^n C \sum_{j=1}^n \prod_{t=1, t \neq j}^n \frac{-\gamma_t^2}{\gamma_j^2 - \gamma_t^2} \xi_{\bar{\sigma}, \nu + \delta_j}.$$

Here $C = \frac{C_n(\mu_1, \dots, \mu_{n-1})}{\prod_{i=1}^{n-1} (\mu_i + i - 1)}$ (see the definition 5).

A base in a \mathfrak{o}_{2n+1} -module V with the highest weight $(\lambda_1, \dots, \lambda_n)$ is constructed inductively by n using the equality $V = \sum_{\mu} V_{\mu}^+ \otimes V(\mu)$, where $V(\mu)$ is a \mathfrak{o}_{2n-1} -representation with the highest weight μ . The result is the following.

Base vectors of V are indexed by tables Λ of type

$$\sigma_n, \lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,n}$$

$$\begin{aligned}
& \lambda'_{n,1}, \lambda'_{n,2}, \dots, \lambda'_{n,n} \\
& \sigma_{n-1}, \lambda_{n-1,1}, \lambda_{n-1,2}, \dots, \lambda_{n-1,n-1} \\
& \dots \\
& \sigma_1, \lambda_{11} \\
& \lambda'_{11}
\end{aligned}$$

The restrictions on these numbers are the following:

1. $\lambda_{ni} = \lambda_i$
2. $\sigma_k = 0, 1$
3. The equalities hold:
$$\lambda'_{k1} \geq \lambda_{k1} \geq \lambda'_{k2} \geq \dots \geq \lambda'_{k,k-1} \geq \lambda'_{kk} \geq \lambda_{kk} \text{ when } k = 1, \dots, n.$$

$$\lambda'_{k1} \geq \lambda_{k-1,1} \geq \lambda'_{k2} \geq \dots \geq \lambda'_{k,k-1} \geq \lambda_{k-1,k-1} \geq \lambda'_{kk} \text{ when } k = 2, \dots, n.$$
4. If $\lambda'_{k1} = 0$, then $\sigma_k = 0$.

Derive formulae for the action of the pfaffian $PfF_{\hat{n}}$. Write the equality $V = \sum_{\mu} V_{\mu}^{+} \otimes V(\mu)$. From one hand the action of the pfaffian on V_{μ}^{+} is already described. From the other hand the pfaffian commutes with \mathfrak{o}_{2n-1} . Thus the action on $V = \sum_{\mu} V_{\mu}^{+} \otimes V(\mu)$ is written as $\sum_{\mu} (PfF_{\hat{n}}|_{V_{\mu}^{+}}) \otimes id$. Hence the pfaffian changes only the two upper rows of the table Λ according to the rule described above.

Write the table Λ as $(\sigma, \lambda, \nu, \Lambda')$, where $\sigma = \sigma_n$, $\nu = \{\lambda_n\}$, $\nu = \{\lambda'_n\}$ and Λ' is the rest part of the table Λ . The base vector corresponding to a table Λ we denote as ξ_{Λ} or $\xi_{\sigma, \nu, \Lambda'}$.

The following theorem is proved

Theorem 2. *On the vector ξ_{Λ} the pfaffian $PfF_{\hat{n}}$ acts as follows.*

Let $\Lambda = (\sigma, \lambda, \nu, \Lambda')$, where σ, λ is first row of Λ , λ' is the second row of Λ and Λ' is the rest part of Λ .

If $\sigma = 0$, then

$$PfF_{\hat{n}}\xi_{\sigma, \nu, \Lambda'} = C\xi_{\sigma, \nu, \Lambda'}.$$

If $\sigma = 1$, then

$$PfF_{\hat{n}}\xi_{\sigma, \nu, \Lambda'} = (-1)^n C \sum_{j=1}^n \prod_{t=1, t \neq j}^n \frac{-\gamma_t^2}{\gamma_j^2 - \gamma_t^2} \xi_{\sigma, \nu + \delta_j, \Lambda'}.$$

Here $C = \frac{C_n(\lambda_{n-1,1}, \dots, \lambda_{n-1,n-1})}{\prod_{i=1}^{n-1} (\lambda_{n-1,i} + i - 1)}$ (see the definition 5).

7 Appendix. Pfaffians and tensor representations.

In the Appendix an action of a pfaffian in an irreducible tensor representation is investigated. Base vectors of such a representation are encoded by orthogonal Young tableaux [21]. In this section at first an action on tensor products of vectors of the standard representation is calculated (propositions 6,7,8,9). Then

Theorem 4 giving some information about the action on the base vectors defined by Young tableaux is proved. In this theorem an image of a base vector is expressed as a linear combination of not necessarily orthogonal tableaux. So this theorem does not produce a formulae of an action of a pfaffian in the bases formed by orthogonal tableaux.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a highest weight of a representation, in this section it is suggested to be integer. To the highest weight λ there corresponds a Young diagram. Let V be a standard representation of \mathfrak{o}_N . Denote by e_i , $i \in \{-n, \dots, n\}$ unit base vectors of V .

In the space $V^{\otimes m}$ there exists a subspace $V^{[m]}$ of traceless tensors. A tensor is traceless if for each pair of indices $1 \leq p < q \leq n$ it belongs to the kernel of all contractions $V^{\otimes n} \rightarrow V^{\otimes(n-2)}$, given by the formulae $v_1 \otimes \dots \otimes v_n \mapsto (v_p, v_q)v_1 \otimes \dots \otimes \widehat{v_p} \otimes \dots \otimes \widehat{v_q} \otimes \dots \otimes v_n$, where $(v_p, v_q) = \delta_{p,-q}$ is a scalar product corresponding to the form G (see sec. 2).

Denote as \mathbb{S}_λ a representation obtained from $V^{\otimes(\sum \lambda_i)}$ by applying the Young symmerizer c_λ , corresponding to the Young diagramm λ .

Theorem 3. (see § 19.5 in [16]) *Let V be a standart representation of \mathfrak{o}_N . The representation $\mathbb{S}_{[\lambda]} := V^{[\sum \lambda_i]} \cap \mathbb{S}_\lambda V$ is irreducible and has the highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$*

To every Young tableau (a Young diagramm filled by numbers) there corresponds a vector v_T in $\mathbb{S}_{[\lambda]}$. To define it let us enumerate places of the Young diagram by the numbers $1, \dots, m$, where $m = \lambda_1 + \dots + \lambda_n$.

Definition 6. Let in the tableau T on the place i stand the number t_i . Take the tensor $e_{t_1} \otimes \dots \otimes e_{t_m}$ and apply to it the Young symmeritizer c_λ corresponding to the diagram. Take the projection of $c_\lambda(e_{t_1} \otimes \dots \otimes e_{t_m})$ to the space of traceless tensors. Denote the resulting tensor as v_T .

The tensors v_T are not linearly independent. To obtain independent tensors v_T one must take v_T corresponding only to the so called orthogonal Young tableaux.

Let us find an action of a pfaffian PfF_I , $|I| = k$ on the vectors v_T of a \mathfrak{o}_N -representation, given by Young tableaux.

It is done in several steps. At first step the action on the vectors e_r of standard representation is described. Then the action on the vectors $e_{r_2} \otimes e_{r_4} \dots \otimes e_{r_t}$, where $t < \frac{k}{2}$ is considered. Then the cases $t = \frac{k}{2}$ and $t > \frac{k}{2}$ are considered. Using these formulaes the action on v_T is described.

Proposition 5. *On the base vectors $e_{-2}, e_{-1}, e_0, e_1, e_2$ of the standard representation of \mathfrak{o}_5 the pfaffians PfF_I where $|I| = 4$ act as zero operators.*

Proof. The proposition is proved by direct calculation using the formulaes, where $a \star b = \frac{1}{2}(ab + ba)$

$$\begin{aligned} PfF_{\widehat{-2}} &= F_{0-1} \star F_{-21} - F_{-1-1} \star F_{-20} + F_{-2-1} \star F_{-10} \\ PfF_{\widehat{-1}} &= F_{0-2} \star F_{-21} - F_{-1-2} \star F_{-20} + F_{-2-2} \star F_{-10} \end{aligned}$$

$$\begin{aligned}
PfF_0 &= F_{1-2} \star F_{-21} - F_{-1-2} \star F_{-2-1} + F_{-2-2} \star F_{-1-1} \\
PfF_1 &= F_{1-2} \star F_{-20} - F_{0-2} \star F_{-2-1} + F_{-2-2} \star F_{0-1} \\
PfF_2 &= F_{1-2} \star F_{-10} - F_{0-2} \star F_{-1-1} + F_{-1-2} \star F_{0-1}
\end{aligned}$$

□

Prove an analog of the previous statement in an arbitrary dimension

Proposition 6. *On the base vectors e_{-n}, \dots, e_n of the standard representation of \mathfrak{o}_N the pfaffians PfF_I for $|I| > 2$ act as zero operators.*

Put $q = 4$, $p = k - 4$ in Lemma 2. One has

$$PfF_I e_j = \sum_{I' \sqcup I'' = I, |I'| = k-4, |I''| = 4} \frac{\left(\frac{p}{2}\right)! \left(\frac{q}{2}\right)!}{\left(\frac{k}{2}\right)!} (-1)^{(I' I'')} PfF_{I'} PfF_{I''} e_j.$$

If $j \notin I''$, then obviously $PfF_{I''} e_j = 0$. If $j \in I''$, then using Proposition 5 one also obtains $PfF_{I''} e_j = 0$.

Let us find an action of a pfaffian of the order k on a tensor product of $< \frac{k}{2}$ vectors, that is on a tensor product $e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t}$, where $t < k$.

Proposition 7. *$PfF_I e_{r_2} \otimes e_{r_4} \dots \otimes e_{r_t} = 0$ where $t < k$*

Proof. The following formulae takes place $\Delta PfF_I = \sum_{I' \sqcup I'' = I} (-1)^{(I' I'')} PfF_{I'} \otimes PfF_{I''}$ (Lemma 4).

By definition one has $PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = (\Delta^k PfF_I) e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k}$. Since $t < k$, the comultiplication $\Delta^k PfF_I$ contains only summands in which on some place the pfaffian stands whose indexing set I satisfies $|I| \geq 4$ (Lemma 4). From Proposition 5 it follows that every such a summand acts as a zero operator. □

Find an action of a pfaffian of the order k on a tensor product of $\frac{k}{2}$ vector, that is on the tensor product $e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k}$.

Proposition 8. *If $\{r_2, r_4, \dots, r_k\}$ is not contained in I , then $PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = 0$.*

Otherwise take a permutation γ of I , such that $(\gamma(i_1), \gamma(i_2), \dots, \gamma(i_k)) = (r_1, r_2, r_3, \dots, r_{k-1}, r_k)$. Then

$$PfF_I e_{r_2} \otimes \dots \otimes e_{r_k} = (-1)^\gamma (-1)^{\frac{k(k-1)}{2}} \sum_{\delta \in \text{Aut}(r_1, r_3, \dots, r_{k-1})} (-1)^\delta e_{-\delta(r_1)} \otimes e_{-\delta(r_3)} \otimes \dots \otimes e_{-\delta(r_{k-3})} \otimes e_{-\delta(r_{k-1})}.$$

Proof. By definition one has

$$PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = (\Delta^k PfF_I) e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k}.$$

Applying many times the formulae for comultiplication one obtains

$$\Delta^{\frac{k}{2}} PfF_I = \sum_{I^1 \sqcup \dots \sqcup I^k} (-1)^{(I^1 \dots I^k)} PfF_{I^1} \otimes \dots \otimes PfF_{I^k}.$$

Using Proposition 7 one gets that, only the summands for which $|I^j| = 2, j = 1, \dots, k$ are nonzero operators.

Hence the summation over divisions can be written in the following manner.

$$PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = \frac{1}{2^{\frac{k}{2}}} \sum_{\sigma \in S_k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} \otimes \dots \otimes F_{-\sigma(i_{k-1})\sigma(i_k)} (e_{r_2} \otimes \dots \otimes e_{r_k}) = \frac{1}{2^{\frac{k}{2}}} \sum_{\sigma \in S_k} (-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} e_{r_2} \otimes \dots \otimes F_{-\sigma(i_{k-1})\sigma(i_k)} e_{r_k}.$$

Consider the expression $F_{-\sigma(i_1)\sigma(i_2)} e_{r_2}$. This is $e_{-\sigma(i_1)}$ if $\sigma(i_2) = r_2$, this is $-e_{-\sigma(i_2)}$ if $\sigma(i_1) = r_2$ and zero otherwise. Thus the summand is nonzero only if the permutation σ satisfies the following condition. In each pair $(\sigma(i_{2t-1}), \sigma(i_{2t}))$ either $\sigma(i_{2t-1}) = r_{2t}$ or $\sigma(i_{2t}) = r_{2t}$.

Show that one can consider only the permutations σ such that $\sigma(i_{2t}) = r_{2t}$, that is the permutations of type $(\sigma(i_1), r_2, \sigma(i_2), r_3, \dots, \sigma(i_{k-1}), r_k)$. But when only summands corresponding to such permutations are considered one must multiply the resulting sum on $2^{\frac{k}{2}}$.

It is enough to prove that the permutations $\sigma = (\sigma(i_1), \sigma(i_2) = r_2, \sigma(i_3), \dots, \sigma(r_k))$ and $\sigma' = (\sigma(i_2) = r_2, \sigma(i_1), \sigma(i_3), \dots, \sigma(r_k))$ give the same input.

Remind that the input for σ is

$$(-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} e_{r_2} \otimes \dots \otimes F_{-\sigma(i_{k-1})\sigma(i_k)} e_{r_k}.$$

One has from one hand that $F_{-\sigma(i_1)\sigma(i_2)} e_{r_2} = e_{-\sigma(i_1)}$ and from the other hand $F_{-\sigma'(i_1)\sigma'(i_2)} e_{r_2} = -e_{-\sigma'(i_2)} = -e_{-\sigma(i_1)}$. Also one has $(-1)^\sigma = -(-1)^{\sigma'}$. Thus the inputs corresponding to σ and σ' are the same.

Hence one can consider the only the permutations σ of type $(\sigma(i_1), r_2, \sigma(i_2), r_3, \dots, \sigma(i_{k-1}), r_k)$ but multiplying the resulting sum on $2^{\frac{k}{2}}$.

For the permutation σ of type $(\sigma(i_1), r_2, \sigma(i_2), r_3, \dots, \sigma(i_{k-1}), r_k)$ using the definition of γ one gets

$$(-1)^\sigma F_{-\sigma(i_1)\sigma(i_2)} e_{r_2} \otimes \dots \otimes F_{-\sigma(i_{k-1})\sigma(i_k)} e_{r_k} = (-1)^{(\sigma(i_1)r_2, \dots, \sigma(i_{k-1})r_k)} e_{-\sigma(i_1)} \otimes \dots \otimes e_{-\sigma(i_k)} = (-1)^{\frac{k(k-1)}{2}} (-1)^\gamma (-1)^\delta e_{-\delta(r_1)} \otimes \dots \otimes e_{-\delta(r_{k-1})}.$$

Here δ is a permutation of the set $\{r_1, r_3, \dots, r_{k-3}, r_k\}$.

The equality $(-1)^{\frac{k(k-1)}{2}} (-1)^\delta (-1)^\gamma = (-1)^\sigma$ was used.

Taking the summation over all permutations δ , one gets

$$PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_k} = (-1)^{\frac{k(k-1)}{2}} (-1)^\gamma \sum_{\delta \in \text{Aut}(r_1, \dots, r_{k-1})} (-1)^\delta e_{-\delta(r_1)} \otimes \dots \otimes e_{-\delta(r_{k-1})}.$$

□

Finally from the formula $PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t} = (\Delta^t PfF_I) e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t}$, as in the proof of Proposition 8, one gets the formulae of the action on an arbitrary tensor $e_{r_2} \otimes \dots \otimes e_{r_t}$.

Proposition 9. $PfF_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t} = \sum_{\{j_2, j_4, \dots, j_k\} \subset \{2, 4, \dots, t\}} Pf^{j_2, j_4, \dots, j_t} F_I e_{r_2} \otimes e_{r_4} \otimes \dots \otimes e_{r_t}$. Here $Pf^{j_2, j_4, \dots, j_k} F_I$ acts on the tensor multiples with numbers j_2, j_4, \dots, j_k . It's action is described by Proposition 8

Find the action of the pfaffian PfF_I on a base vector v_T .

By definition the action PfF_I on v_T is constructed as follows.

1. To the tensor product $e_{t_1} \otimes \dots \otimes e_{t_m}$ the Young symmetrizer c_λ is applied.
2. The projection on the space of traceless tensors is applied.
3. The pfaffian PfF_I is applied.

Change the order of operations.

Since the Young symmetrizer and the projection on the space of traceless tensors commute with the action of \mathfrak{o}_N (and hence with PfF_I) one can first apply the pfaffian, then the symmetrizer and finally the projection.

Using the propositions 9, 8 one gets the following theorem.

Theorem 4. *The image of $PfF_I v_T$ can be found as follows*

1. *In all possible ways in the tableau T choose $\frac{k}{2}$ places in the Young tableau, such that on them different indices $r_2, r_4, \dots, r_k \in I$ stand. Find a permutation γ of I , such that $(\gamma(i_1), \dots, \gamma(i_k)) = (r_1, r_2, r_3, \dots, r_{k-1}, r_k)$.*

Replaced indices r_2, r_4, \dots, r_k in all possible ways onto the indices $-r_1, -r_3, \dots, -r_{k-1}$.

Take the alternative sum of tableaux $\sum \pm T'$, the tableau in this sum are obtained from the initial one by placing $-r_1, -r_3, \dots, -r_{k-1}$ on the chosen places. The places in the tableau are ordered and the sign is defined by the placing of indices $-r_1, -r_3, \dots, -r_{k-1}$ on these places. The resulting sum is multiplied by $(-1)^\gamma (-1)^{\frac{k(k-1)}{2}}$.

If it is not possible to choose in T places in which stand different indices $r_2, r_4, \dots, r_k \in I$, then $PfF_I v_T = 0$.

2. *For every T' the tensor $v_{T'}$ is constructed.*
3. *The sum $\sum_{T'} v_{T'}$ is taken. The result is $PfF_I v_T$.*

References

- [1] A.W.M. Dress, W. Wenzel. A simple proof of an identity concerning pfaffians of skew symmetric matrices. Adv. Math, 112, 120-134 (1995)
- [2] M. Ishikawa, S. Okada, M. Wakayama. Applications of Minor-Summation Formula I. Littlewood's Formulas. J. Algebra 183, 1, 1996, 193-216
- [3] M. Ishikawa, M. Wakayama. Applications of Minor-Summation Formula II. Pfaffians and Schur Polynomials. J. Comb. Theory, Series A 88, 1, 1999, 136-157
- [4] M. Ishikawa, M. Wakayama. Applications of minor summation formula III, Plucker relations, lattice paths and Pfaffian identities. J. Comb. Theory, Series A 113, 1, 2006, 113-155

- [5] T Hashimoto. A Central Element in the Universal Enveloping Algebra of Type D_n via Minor Summation Formula of Pfaffians. *Journal of Lie Theory*, 18 (2008), No. 3, 581–594
- [6] S. Goshen, H.J. Lipkin. On the application of the group $Sp(4)$ or $R(5)$ to nuclear structure. *Spectroscopy and group theoretical methods in physics*, North-Holland Publ. Co. Amsterdam, 1968, 245-273.
- [7] D. J. Rowe¹, R. Le Blanc², and K. T. Hecht³, Vector coherent state theory and its application to the orthogonal groups *J. Math. Phys.* 29, 287 (1988); doi:10.1063/1.528066 (18 pages)
- [8] D. J. Rowe, Vector coherent state representations of $so(5)$ in an $so(3)$ basis, *J. Math. Phys.* 35, 3163 (1994); doi:10.1063/1.530459 (15 pages)
- [9] P. S. Turner¹, D. J. Rowe¹, and J. Repka² Vector coherent state theory of the generic representations of (5) in an (3) basis *J. Math. Phys.* 47, 023507 (2006); doi:10.1063/1.2162332 (25 pages)
- [10] A I Molev Weight bases of Gelfand-Tsetlin type for representations of classical Lie algebras *J. Phys. A: Math. Gen.* 2000, 33 4143 doi: 10.1088/0305-4470/33/22/316
- [11] A. I. Molev, A Basis for Representations of Symplectic Lie Algebras, *Communications in Mathematical Physics* Volume 201, Number 3, 591-618, DOI: 10.1007/s002200050570
- [12] A.I. Molev, Yangians and transvector algebras, *Discrete Mathematics* Volume 246, Issues 1-3, 6 March 2002, Pages 231-253
- [13] A. I. Molev. A weight basis for representations of even orthogonal Lie algebras *Adv. Studies in Pure Math.* 28 (2000), 223–242
- [14] D.P. Zhelobenko. UMN.
- [15] N. Bourbaki. *Elements de Mathematique. Groupes et algebres de Lie*, chapters 7,8, Springer, 2006.
- [16] W. Fulton, J. Harris. *Representation theory, the first course*, 1991, Springer, New-York.
- [17] D P Zhelobenko Extremal projectors and generalized Mickelson algebras over reductive Lie algebras. *Mathematics of the USSR-Izvestiya* (1989), 33(1):85.
- [18] A. Molev. *Yangians and classical Lie algebras*, 2007, AMS, *Mathematical Surveys and Monographs*, vol. 143,
- [19] A.I. Molev. Gelfand-Tsetlin bases for classical Lie algebras. *Handbook of Algebra* Volume 4, 2006, Pages 109-170

- [20] I. Umeda, Capelli elements for the orthogonal Lie algebras, J. Lie Theory, V 10, 2000, 463-489
- [21] R.A. Proctor. Young Tableaux, Gelfand Patterns, and Branching Rules for Classical Groups. Journal of Algebra Volume 164, Issue 2, 1 March 1994, Pages 299-360